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Parameter space reduction criteria to search for synchronization domains in coupled discrete systems

J. Escalona,[1] T. E. Govindan,[1,2] G. Martínez-Mekler,[1,3] and P. Parmananda[1]
[1]*Facultad de Ciencias, UAEM, Cuernavaca 62210, Morelos, Mexico*
[2]*Centro de Investigación en Matemáticas, Guanajuato 36000, Guanajuato, Mexico*
[3]*Centro de Ciencias Físicas, UNAM, Cuernavaca 62210, Morelos, Mexico*

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Based on analytical considerations, we introduce criteria that enable us to encapsulate the parameter domains for which chaotic synchronization in linearly coupled map systems may be attained. Our aim is to provide means to readily determine parameter regions which preclude synchronization. This results in a significant reduction of parameter space that one needs to explore. Our findings hold for both identical and quasi-identical (small parameter mismatch) maps subjected to unidirectional and bidirectional coupling. As a testing ground we present numerical calculations for the logistic and cubic maps which validate the predictive capability of our approach. Our main contribution relies on the applicability of one of our criteria to experimental situations. Since in real life it is almost impossible to construct two truly identical systems, the results for quasi-identical maps are of particular relevance.

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Interest in synchronization is easily motivated due to its ubiquitous nature and its relevance to a number of physical [1–3], chemical [4], biological [5], and ecological systems [6]. The first documented observation of synchronization dates back to Huygens and his pendulum clocks in 1665. In the last decade there has been considerable interest generated in chaotic synchronization since the work of Pecora and Carroll [7,8]. Apart from being an interesting scientific problem, chaotic synchronization is deemed to be of technological importance as in the field of secure communications [9]. Recent books [10,11] and a review article [12] give an exhaustive historical background and document advances in the field of synchronization.

In this Rapid Communication we present analytical considerations that enable one to obtain parameter bounds on the chaotic synchronization domains for identical and quasi-identical systems (small parameter mismatch) under unidirectional and bidirectional couplings. Furthermore, one of the criteria introduced does not require explicit knowledge of the mapping function in order to predict bounds on the synchronization region. Since in experiments one can convert a continuous data stream to a discrete set using return maps, our approach could allow one to eliminate large domains of parameter space where synchronization is precluded and focus on the exploration of parameter regions of interest (where synchronization may be attained).

Two unidirectionally coupled maps defined by the following set of equations:

$$x_{n+1} = f_a(x_n), \tag{1}$$

$$y_{n+1} = f_a(y_n) - \gamma(x_n - y_n), \tag{2}$$

are considered for our discrete dynamics. Here a is the bifurcation parameter of the map (there may be a set of such parameters), and γ is the coupling strength. Due to the coupling term in the slave system [Eq. (2)], at times the system may exhibit blow-out instabilities [13]. To eliminate such problems in our numerical calculations, an artificial reset of the slave dynamics to randomly chosen values within the chaotic attractor is imposed whenever y_n crosses a predetermined threshold in state space. However, it needs to be emphasized that the artificial reset is a transient feature that is only required for a finite number of iterations after the coupling is switched "ON". Therefore it has no effect on the asymptotically synchronized chaotic trajectories. Since the criteria developed to predict the domains of synchronization entail using asymptotic behavior, they are independent of the resetting procedure.

We define the error function $e_n = x_n - y_n$. Subtracting Eq. (2) from Eq. (1), and subsequent factorization, yields

$$e_{n+1} = [G_a(x_n, y_n) + \gamma]e_n, \tag{3}$$

where

$$G_a(x_n, y_n) = \frac{f_a(x_n) - f_a(y_n)}{e_n}. \tag{4}$$

Our condition for reducing the parameter space to be explored is that γ satisfy the inequality

$$-1 - G_a^{\max} < \gamma < 1 - G_a^{\min}, \tag{5}$$

where $G_a^{\min} \equiv \min\{G_a(x_n, y_n)$, given a, and for all (x_n, y_n) with n sufficiently large$\}$ and $G_a^{\max} \equiv \max\{G_a(x_n, y_n)$, given a, and for all (x_n, y_n) with n sufficiently large$\}$.

If γ is chosen outside the range determined by Eq. (5) it is impossible to attain synchronization. The proof of this statement follows.

First, consider the right-hand side of inequality Eq. (5), which determines an upper bound condition on γ

$$\gamma < 1 - G_a^{\min}. \tag{6}$$

If Eq. (6) does not hold, i.e., $\gamma \geq 1 - G_a^{\min}$, then by definition of $G_a^{\min}$, we have that

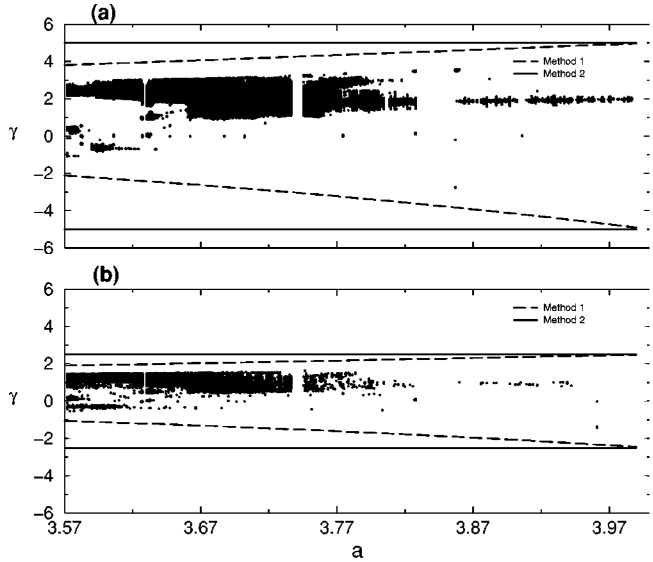

FIG. 1. The region denoted by black dots corresponds to the extent of parameter space where total synchronization is attained (numerically) for logistic maps subjected to (a) unidirectional and (b) bidirectional couplings. The synchronization domains shown correspond to asymptotic behavior. We present only the parameter region where the value of the map parameter a leads to chaotic dynamics. The values of the control parameter corresponding to periodic windows in the bifurcation diagram were intentionally removed. Analytical bounds predicted by using method 1 and method 2 are also shown.

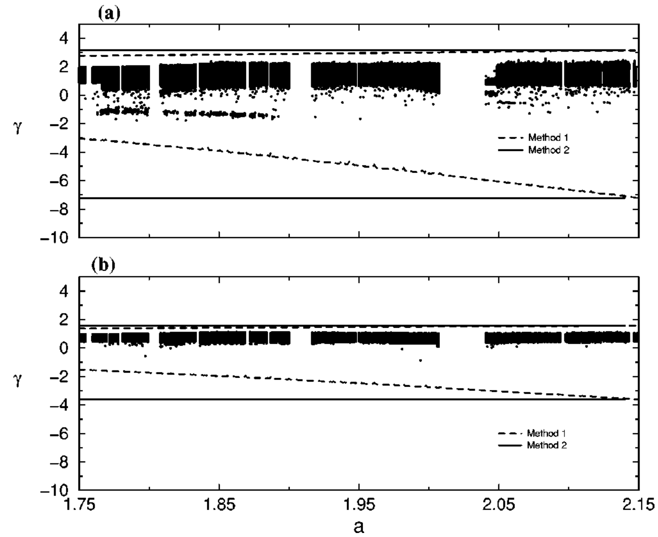

FIG. 2. The region denoted by black dots corresponds to the extent of parameter space where total synchronization is attained (numerically) for cubic maps subjected to (a) unidirectional and (b) bidirectional couplings. The synchronization domains shown correspond to asymptotic behavior. We present only the parameter region where the value of the map parameter a leads to chaotic dynamics, the other parameter for the map is fixed at $b=0.5$. The values of the control parameter corresponding to periodic windows in the bifurcation diagram were intentionally removed. Analytical bounds predicted using method 1 and method 2 are also shown.

$$\gamma + G_a(x_n,y_n) \geqslant 1, \tag{7}$$

for all (x_n, y_n) with n sufficiently large.

Next consider the left-hand side of inequality Eq. (5), which determines a lower bound condition on γ

$$\gamma > -1 - G_a^{\max}. \tag{8}$$

If Eq. (8) does not hold, i.e., $\gamma \leqslant -1 - G_a^{\max}$, then by definition of $G_a^{\max}$ we have

$$-\gamma - G_a(x_n,y_n) \geqslant 1 \tag{9}$$

for all (x_n, y_n) with n sufficiently large.

Notice that the conditions expressed by Eqs. (7) and (9) are exclusive, in the sense that a given γ can only satisfy one of them. Furthermore, Eq. (7) implies that $G_a(x_n, y_n) + \gamma > 0$, while Eq. (9) ensures $G_a(x_n, y_n) + \gamma < 0$. Thus we have that the values of γ, which either violate Eq. (6) or violate Eq. (8), satisfy, for all (x_n, y_n) with n sufficiently large, the condition expressed by the inequality

$$|G_a(x_n,y_n) + \gamma| \geqslant 1, \tag{10}$$

which, according to Eq. (3), impedes synchronization. Hence, we have shown unequivocally that outside the range of values for γ determined by Eq. (5), synchronization is precluded. However, it should be emphasized that if γ does satisfy the inequalities of Eq. (5) synchronization is not guaranteed.

Notice that since in Eq. (5) we invoke the minimum and maximum over all the values of $G_a(x_n, y_n)$ for n sufficiently large, it is a condition on the asymptotic dynamics. A further reduction on the parameter space can be obtained if the inequality Eq. (5) is relaxed for a set of instances of measure 0 as $n \to \infty$.

In Figs. 1(a) and 2(a) we show, as a testing ground, that Eq. (5) provides a *criterium* for parameter space reduction susceptible to synchronization for the chaotic dynamics of the logistic $[x_{n+1} = a x_n(1 - x_n)]$ and cubic $(x_{n+1} = b - a x_n + x_n^3)$ maps. For these maps, $G_a(x_n, y_n) = a[1 - (x_n + y_n)]$ and $G_a(x_n, y_n) = (x_n^2 + x_n y_n + y^2) - a$, respectively. The bounds (dashed lines) were determined from the corresponding $G_a^{\min}$ and $G_a^{\max}$. We refer to this procedure as "method 1." In the figures we also show the numerically calculated synchronization regions for the two maps subjected to unidirectional coupling.

Our main interest in Eq. (5) is its relevance to experiments where one assumes the nonavailability of the functional form of the map. Notice that when $\gamma = 0$, i.e., for the case of uncoupled dynamics, Eq. (4) is given as

$$G_a\,(x_n,y_n)\big|_{\gamma=0} = \frac{x_{n+1} - y_{n+1}}{x_n - y_n}, \tag{11}$$

which is an expression that can be calculated directly from the experimental data. Using Eq. (11), the extremal values for each a of the function $G_a\,(x_n, y_n)|_{\gamma=0}$ can be determined. Subsequently, one may look for constants C_1 and C_2 that satisfy the following conditions:

$$C_1 \equiv \max\{ G_a^{\max}|_{\gamma=0} \text{ for all } a \text{ (for chaotic region)}\}, \quad (12)$$

$$C_2 \equiv \min\{ G_a^{\min}|_{\gamma=0} \text{ for all } a \text{ (for chaotic region)}\}. \quad (13)$$

It is reasonable to assume that the function G for the chaotic dynamics attains less extreme values, largest and smallest, for the coupled case ($\gamma \neq 0$) than for the uncoupled scenario ($\gamma=0$). Using Eq. (5), the appropriate C_1 and C_2 would then satisfy the inequality

$$-1 - C_1 \leqslant -1 - G_a^{\max} < \gamma < 1 - G_a^{\min} \leqslant 1 - C_2. \quad (14)$$

Equation (14) provides us with a *second criterium* for parameter space reduction for synchronization which is particularly suitable for experiment. One can envisage two different scenarios for Eq. (14).

(i) If the search for chaotic synchronization is desired for a fixed value of the bifurcation parameter a, one can estimate G_a from the experimental data and select its maximum and minimum values.

(ii) If the search for chaotic synchronization is desired for a finite domain of parameters an experimental bifurcation diagram needs to be constructed. One estimates C_1 and C_2 using the bifurcation parameters a that yield the most extremal values of the function G_a.

We should emphasize that for both the above two scenarios one calculates G_a with $\gamma=0$ using the dynamical evolution of a single system initialized at different initial conditions. This property may be of great advantage for particular experimental setups.

The bounds obtained for the logistic and cubic maps by means of a numerical implementation of the second scenario, which we label as "method 2," are shown in the figures as solid lines. A good set of statistics on the calculated values of G_a using different sets of initial conditions was required for the proper estimation of C_1 and C_2. The values $-1-C_1$ and $1-C_2$ delimit the parameter space available for synchronization. A comparison with the numerically calculated synchronization regions, also shown in the figures, indicates that our second analytic criterium provides a reasonable estimate for the domains of chaotic synchronization.

Our analysis can be easily extended to maps with bidirectional coupling given by

$$x_{n+1} = f_a(x_n) + \gamma(x_n - y_n), \quad (15)$$

$$y_{n+1} = f_a(y_n) - \gamma(x_n - y_n), \quad (16)$$

where a is the bifurcation parameter of the map and γ is the coupling strength. Similar to the unidirectional case, an artificial reset of the dynamics was imposed whenever x_n or y_n crossed a predetermined threshold in state space.

Following the same procedure as for the unidirectional case yields the following bounds for the bidirectional case:

$$\frac{-1 - C_1}{2} \leqslant \frac{-1 - G_a^{\max}}{2} < \gamma < \frac{1 - G_a^{\min}}{2} \leqslant \frac{1 - C_2}{2}. \quad (17)$$

Figure 1(b) and Fig. 2(b) show the numerically calculated synchronization regions for the two maps subjected to bidirectional coupling. The bounds obtained using the two methods described before are plotted and again are in agreement with numerical simulations.

Our line of reasoning can be applied to the case of two unidirectionally coupled nonidentical (parameter mismatch) maps defined by the following set of equations:

$$x_{n+1} = f_a(x_n), \quad (18)$$

$$y_{n+1} = f_{a+\Delta a}(y_n) - \gamma(x_n - y_n), \quad (19)$$

a and $a+\Delta a$ are the parameters of the master and the slave maps, respectively, and γ is the coupling strength. Resetting of the slave dynamics is performed as for the identical maps. Our method for nonidentical maps is valid for maps where a function exists $g(y_n)$ such that $f_{a+\Delta a}(y_n)=f_a(y_n)+g(y_n)\Delta a$. However, if such a factorization is not possible and $\Delta a \ll 1$, it suffices to truncate the expansion of $f_{a+\Delta a}$ in Δa at the linear term of Δa. Following the protocol established for the identical maps and using $G_a(x_n, y_n)$ defined in Eq. (4) yields

$$e_{n+1} = [G_a(x_n,y_n) + \gamma]e_n - g(y_n)\Delta a. \quad (20)$$

Taking the absolute value of the preceding expression and using the triangle inequality gives

$$|e_{n+1}| \leqslant |G_a(x_n,y_n) + \gamma||e_n| + |g(y_n)||\Delta a|. \quad (21)$$

Assuming quasisynchronization (where two systems are considered quasisynchronized if for large n, $|x_n - y_n| < \epsilon$ where $\epsilon \ll 1$), in the asymptotic limit $|e_{n+1}| \approx |e_n|$. Taking this last relation as an equality, multiplying both sides of Eq. (21) by $|e_n|$, and rearranging yields

$$|e_n|\{|e_n| - |G_a(x_n,y_n) + \gamma||e_n| - |g(y_n)||\Delta a|\} \leqslant 0. \quad (22)$$

Since $|e_n| \geqslant 0$, the term $|e_n| - |G_a(x_n,y_n)+\gamma||e_n| - |g(y_n)||\Delta a|$ should be less than or equal to zero. A little algebra gives

$$|e_n| \leqslant \frac{|g(y_n)||\Delta a|}{1 - |G_a(x_n,y_n) + \gamma|}. \quad (23)$$

Now we assume that the right-hand side of the preceding inequality satisfies

$$\frac{|g(y_n)||\Delta a|}{1 - |G_a(x_n,y_n) + \gamma|} \leqslant \epsilon, \quad (24)$$

then

$$-1 - G_a(x_n,y_n) + \frac{|g(y_n)||\Delta a|}{\epsilon} \leqslant \gamma \leqslant 1 - G_a(x_n,y_n) - \frac{|g(y_n)||\Delta a|}{\epsilon}. \quad (25)$$

This expression furnishes upper and lower bounds of the quasisynchronization region for nonidentical maps since it relies on the inequality Eq. (24) which, if violated, impedes quasisynchronization. It can be easily verified that the bounds given by Eq. (25) reduce to the bounds given by Eq. (5) for $\Delta a=0$.

Similar to the previous results, the bounds reduce by a factor of two for the bidirectional coupling of quasi-identical maps. It is evident from Eq. (25) that the bounds predicted using the analytical considerations for the quasi-identical case ($\Delta a \ll 1$) will be inside the bounds for the identical case (due to the emergence of the $|g(y_n)||\Delta a|/\epsilon$ term). Therefore the bounds of the identical case remain valid for the quasi-synchronization region of the quasi-identical maps for both unidirectional and bidirectional couplings. Another important observation from Eq. (25) is that the larger the parameter mismatch (Δa), the smaller is the quasisynchronization region. This was also confirmed numerically (results not shown).

To summarize, our results indicate that it is possible to predict upper and lower bounds which enclose parameter regions where synchronization may occur. Our approach relies on simple analytic considerations which produce criteria, in the form of parameter relations, for the selection of parameter space regions in which to explore for synchronization. The proposed methodology is valid for synchronization of identical and quasi-identical systems subjected to both unidirectional and bidirectional linear coupling. Moreover, one of the criteria is particularly suitable for online experimental determination and may be implemented on the uncoupled system of chaotic oscillators. The methodology we have presented may constitute an important tool for the experimentalists, allowing them to eliminate parameter domains where total synchronization is precluded.

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